

# Euclidean lattice simulation for the dynamical supersymmetry breaking

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The global supersymmetry is spontaneously broken if and only if the ground-state energy is strictly positive. We propose to use this fact to observe the spontaneous supersymmetry breaking in euclidean lattice simulations. For lattice formulations that possess a manifest fermionic symmetry, there exists a natural choice of a hamiltonian operator that is consistent with a topological property of the Witten index. We confirm validity of our idea in models of the supersymmetric quantum mechanics. We then examine a possibility of a dynamical supersymmetry breaking in the two-dimensional  $\mathcal{N} = (2, 2)$  super Yang-Mills theory with the gauge group  $SU(2)$ , for which the Witten index is unknown. Differently from a recent conjectural claim, our numerical result tempts us to conclude that supersymmetry is not spontaneously broken in this system.

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It is widely believed that supersymmetry is relevant in particle physics beyond the standard model and it is spontaneously broken by some mechanism. If supersymmetry is not spontaneously broken at the tree level of the loop expansion, it remains so to all orders of the loop expansion. There still exists, however, a possibility that supersymmetry is spontaneously broken non-perturbatively. Precise study of such *dynamical supersymmetry breaking* remains elusive because we have no universal framework that defines supersymmetric (especially gauge) theories at a non-perturbative level.

Generally, the Witten index  $\text{Tr}(-1)^F$  [1], where  $F$  is the fermion number operator, provides an important clue. One can infer that the dynamical supersymmetry breaking does not occur in a wide class of supersymmetric models for which the Witten index can be computed to be non-zero. However, the Witten index is not a panacea. There exist physically interesting models for which it is very difficult to determine the Witten index and, in some cases, the index itself might be ill-defined due to a gapless continuous spectrum.

In this letter, we consider a possibility to observe the dynamical supersymmetry breaking in euclidean lattice simulations, in the light of recent developments on lattice formulation of supersymmetric theories [2, 3, 4, 5]. The conceptually clearest way to observe the spontaneous supersymmetry breaking would be to examine the degeneracy of bosons' and fermions' mass spectra in two-point correlation functions. Here, we propose an alternative method that is based on the following fact. The global supersymmetry is spontaneously broken if and only if the ground-state energy is strictly positive [6]. In principle,

therefore, one can judge whether the supersymmetry breaking takes place or not if the ground-state energy can be computed.

Let us first recall that the thermal average of the hamiltonian  $H$  with the inverse temperature  $\beta$  is expressed by the euclidean functional integral as [22]

$$\frac{\text{Tr} H e^{-\beta H}}{\text{Tr} e^{-\beta H}} = \frac{\int_{\text{aPBC}} d\mu H e^{-S}}{\int_{\text{aPBC}} d\mu e^{-S}} \equiv \langle H \rangle_{\text{aPBC}}, \quad (1)$$

where  $S$  is the euclidean (i.e., imaginary-time) action and  $d\mu$  symbolically denotes a functional integral measure. In the right-hand side, the time-period of the system is taken to be  $\beta$ . What is very important in Eq. (1) is the boundary condition in the temporal direction. It must be periodic for all bosonic variables and anti-periodic (aPBC) for all fermionic variables. (For all bosonic variables and for all variables with respect to spatial directions, we always assume the periodic boundary conditions.) In the large imaginary-time or the low-temperature limit  $\beta \rightarrow \infty$ , only the ground state(s) contributes to Eq. (1) and the ground-state energy  $E_0$  is given by

$$E_0 = \lim_{\beta \rightarrow \infty} \langle H \rangle_{\text{aPBC}}. \quad (2)$$

If  $E_0 > 0$ , supersymmetry is spontaneously broken and it is not if  $E_0 = 0$ .

Suppose that in Eq. (1) one uses instead the periodic boundary condition (PBC) for all variables. Then the partition function is proportional to the Witten index  $\mathcal{N}_{\text{PBC}} \int_{\text{PBC}} d\mu e^{-S} = \text{Tr}(-1)^F e^{-\beta H} = \text{Tr}(-1)^F$  [9, 10] and  $\int_{\text{PBC}} d\mu H e^{-S}$  is proportional to the  $\beta$ -derivative of the Witten index, that is *always* zero

$$\mathcal{N}_{\text{PBC}} \int_{\text{PBC}} d\mu H e^{-S} = \text{Tr}(-1)^F H e^{-\beta H} = 0. \quad (3)$$

This independence of the Witten index on a parameter of the theory  $\beta$  is a consequence of the supersymmetry

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algebra [1]. We also note that, when the Witten index is non-zero, Eq. (3) is not invariant under a shift of the origin of the energy  $H \rightarrow H + c$ . With the periodic boundary condition, we thus have (independently of  $\beta$ )

$$\langle H \rangle_{\text{PBC}} \equiv \frac{\int_{\text{PBC}} d\mu H e^{-S}}{\int_{\text{PBC}} d\mu e^{-S}} = \frac{0}{\text{Tr}(-1)^F} \quad (4)$$

and clearly this provides no useful information on the ground-state energy.

In Eq. (1), boundary conditions for realizing the thermal equilibrium explicitly breaks supersymmetry. In Eq. (2), we then observe how the effect of the temperature (that is a conjugate variable to the energy) remains in the zero-temperature limit  $\beta \rightarrow \infty$ . If the effect remains, that is,  $E_0 > 0$  in Eq. (2), we judge that the spontaneous supersymmetry breaking occurs. It is interesting to note that this procedure is quite analogous to a usual way to observe the spontaneous breaking of ordinary symmetries [23].

There are several issues to be clarified to embody the basic formula (2) in euclidean lattice formulation. First, the above argument assumes that a regularization to define the functional integral does not break supersymmetry. Although lattice regularizations are generally irreconcilable with supersymmetry, for theories with extended supersymmetry, it is sometimes possible to set up a lattice regularization that preserves the invariance under a part of supersymmetry transformations [2, 3, 4, 5]. Then, if the spacetime dimension is low enough, one may expect that the invariance under a full set of supersymmetry transformations is restored in the continuum limit. In what follows, we assume this sort of lattice regularization.

Closely related to the above point, we have to properly choose a possible additive constant in the hamiltonian  $H$ . In other words, we have to correctly choose the origin of the energy. This is of course a crucially important point to judge the spontaneous supersymmetry breaking from the positivity of  $E_0$ . A natural prescription to define the hamiltonian is to use the supersymmetry algebra [24]. By the following reason, however, this issue of a “correct” hamiltonian is somewhat delicate in the functional integral formulation based on the lagrangian.

Suppose that the (for simplicity, off-shell) supersymmetry algebra is realized by the transformation law for variables appearing in the continuum lagrangian. This implies that there exists a fermionic transformation  $Q$  such that  $\{Q, \bar{Q}\} = 2i\partial_0$ , where  $\bar{Q}$  is a fermionic transformation conjugate to  $Q$  and  $\partial_0$  is the time derivative. One would then expect from this algebra that the relation  $iQ\bar{Q} = 2H$  holds, where  $\bar{Q}$  is the Noether charge (in field theory, we use the Noether current instead) associated with the transformation  $\bar{Q}$  and  $H$  is the hamiltonian obtained from the lagrangian by the Legendre transformation.

In reality, this relation holds only *up to equations of motion*. Generally one ends up with  $iQ\bar{Q}/2 = H +$  (terms being proportional to equations of motion). The

additional terms, that would be negligible in classical theory, cannot be neglected in general within the functional integral because those terms may give rise to contact terms at a coincident point, i.e., ultraviolet-divergent constants [25].

However, if a lattice formulation one adopts possesses at least one exactly-preserved fermionic symmetry, say the above  $Q$ , it is natural to adopt  $H \equiv iQ\bar{Q}/2$  as the definition of a hamiltonian. First, this structure is suggested from the supersymmetry algebra. Second, this choice has the correct origin of the energy in the sense that it is consistent with the topological property of the Witten index, Eq. (3), that is,

$$\int_{\text{PBC}} d\mu H e^{-S} = \int_{\text{PBC}} d\mu Q \left( \frac{i}{2} \bar{Q} e^{-S} \right) = 0, \quad (5)$$

where we have assumed that the lattice action  $S$  and the integration measure  $d\mu$  are invariant under the  $Q$ -transformation and the integral  $\int_{\text{PBC}} d\mu \bar{Q} e^{-S}$  is finite. As already noted, when the Witten index is non-zero, this property fixes the origin of the energy uniquely. For these reasons, we adopt the  $Q$ -exactness of the hamiltonian,  $H \equiv iQ\bar{Q}/2$ , as a working hypothesis in what follows.

We examine our idea by applying it to a euclidean lattice formulation of the supersymmetric quantum mechanics [6]. The euclidean lattice action of this model can be taken as [12, 13, 14]

$$S = \sum_{x \in \Lambda} \left\{ \frac{1}{2} \partial \phi(x) \partial \phi(x) + \frac{1}{2} (W'(\phi(x)))^2 + \bar{\psi}(x) (\partial + W''(\phi(x))) \psi(x) - \frac{1}{2} F(x)^2 + W'(\phi(x)) \partial \phi(x) \right\}, \quad (6)$$

where  $\Lambda = \{x \in a\mathbb{Z} \mid 0 \leq x < \beta\}$  ( $a$  denotes the lattice spacing),  $\partial$  is the forward difference  $\partial f(x) \equiv f(x+a) - f(x)$  and  $\psi(x=\beta) = +\psi(x=0)$  for PBC and  $\psi(x=\beta) = -\psi(x=0)$  for aPBC. This lattice action is invariant under a lattice counterpart of one of the  $\mathcal{N} = 2$  supersymmetry transformations,  $Q$ , that is defined by  $Q\phi(x) = \psi(x)$ ,  $Q\psi(x) = 0$ ,  $Q\bar{\psi}(x) = F(x) - \partial\phi(x) - W'(\phi(x))$  and  $QF(x) = \partial\psi(x) + W''(\phi(x))\psi(x)$ . The corresponding continuum theory is invariant under also  $\bar{Q}$  and the supersymmetry algebra reads  $Q^2 = \bar{Q}^2 = 0$  and  $\{Q, \bar{Q}\} = -2\partial$ . Corresponding to this  $\bar{Q}$ -symmetry, we have the Noether charge  $\bar{Q}$ . By using a lattice transcription of this Noether charge  $\bar{Q}(x) \equiv -\bar{\psi}(x) (i\partial\phi(x) - iW'(\phi(x))) / a$ , we define the hamiltonian operator by  $H(x) \equiv iQ\bar{Q}(x)/2$ . As elucidated above, this lattice hamiltonian has a correct zero-point energy in the sense of Eq. (5). In the corresponding continuum theory [6], it is well-known that supersymmetry is spontaneously broken if and only if the number of zeros of the function  $W'(\phi)$  is even.

As a definite example, we consider

$$W(\phi(x)) = \frac{1}{2}(am)\phi(x)^2 + \frac{1}{3}(am)^{3/2}\lambda\phi(x)^3, \quad (7)$$

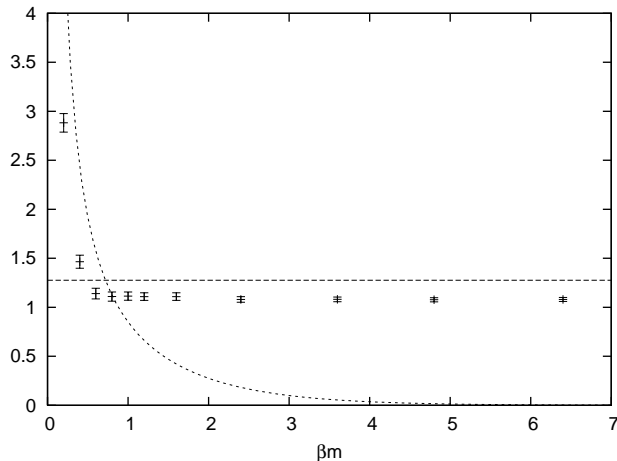


FIG. 1:  $\lim_{a \rightarrow 0} \langle H(x) \rangle_{\text{aPBC}}/m$  as a function of the physical temporal size of the system  $\beta m$ . The errors are only statistical ones. The dotted curve is the analytic expression for the  $\lambda = 0$  case for which supersymmetry is not broken.

where  $m$  is a parameter which has the mass dimension 1. Supersymmetry is dynamically broken in the corresponding continuum theory when  $\lambda \neq 0$ .

Fig. 1 is a result of Monte Carlo simulation for  $\lambda = 10$ . The continuum limit of the expectation value of the hamiltonian with the anti-periodic boundary condition,  $\lim_{a \rightarrow 0} \langle H(x) \rangle_{\text{aPBC}}/m$ , is plotted as a function of the physical temporal size of the system  $\beta m$  [26]. For  $\beta m \gtrsim 1$ , we have  $\lim_{a \rightarrow 0} \langle H(x) \rangle_{\text{aPBC}}/m \simeq 1.1$  [27]. From this, we infer that supersymmetry is spontaneously broken and this is indeed the right answer. We also numerically observed that, in the present system,  $\langle H(x) \rangle_{\text{PBC}}$  is not well-defined, being consistent with  $\text{Tr}(-1)^F = 0$  in the target theory (recall Eq. (4)) [15].

Next, as an example in which supersymmetry is *not* spontaneously broken, we consider

$$W(\phi(x)) = \frac{1}{4}(am)^2 \phi(x)^4. \quad (8)$$

In this case, we numerically observed that both  $\langle H(x) \rangle_{\text{PBC}}$  and  $\langle H(x) \rangle_{\text{aPBC}}$  are well-defined and, in Fig. 2, we plotted the continuum limit of these quantities as a function of  $\beta m$  [28]. The figure shows that, in this case,  $\lim_{a \rightarrow 0} \langle H(x) \rangle_{\text{PBC}}$  is consistent with zero for all temporal sizes (recall Eq. (4); in the target theory  $\text{Tr}(-1)^F = 1$ ) and  $\lim_{a \rightarrow 0} \langle H(x) \rangle_{\text{aPBC}}$  approaches zero as the temporal size of the system is increased. From Eq. (2), we conclude that  $E_0 = 0$  within the error and supersymmetry is not broken.

Having observed that our method works perfectly in the supersymmetric quantum mechanics, we now study the two-dimensional  $\mathcal{N} = (2, 2)$  super Yang-Mills theory by using a lattice formulation proposed in Ref. [16]. (See also Ref. [17].) For this seemingly simple supersymmetric system, the value of the Witten index and whether supersymmetry is spontaneously broken or not are not

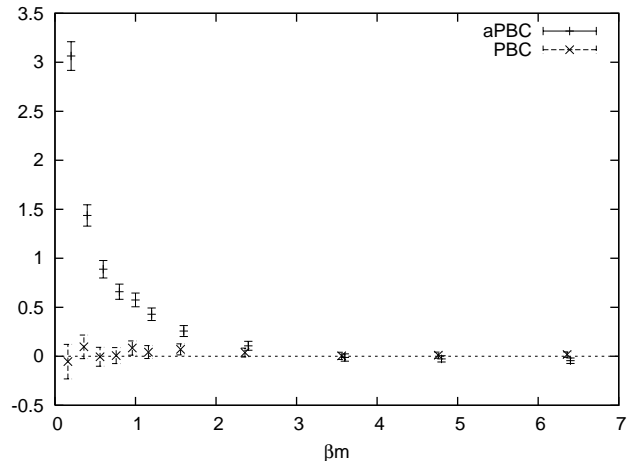


FIG. 2:  $\lim_{a \rightarrow 0} \langle H(x) \rangle_{\text{aPBC}}/m$  and  $\lim_{a \rightarrow 0} \langle H(x) \rangle_{\text{PBC}}/m$ , as a function of the physical temporal size of the system  $\beta m$ . The errors are only statistical ones.

known, due to *non-compact* flat directions of the classical potential energy. In fact, the authors of Ref. [18] conjectured the dynamical supersymmetry breaking in this system with the gauge group  $\text{SU}(N_c)$ .

We numerically studied only the case of the gauge group  $\text{SU}(2)$ . The physical size of our two-dimensional lattice  $\Lambda$  is  $\beta \times L$ ;  $\Lambda = \{x \in a\mathbb{Z}^2 \mid 0 \leq x_0 < \beta, 0 \leq x_1 < L\}$  and the boundary condition of a generic fermionic field  $\psi$  is set to be,  $\psi(x_0 = \beta, x_1) = +\psi(x_0 = 0, x_1)$  for PBC and  $\psi(x_0 = \beta, x_1) = -\psi(x_0 = 0, x_1)$  for aPBC [29].

The point is that the lattice action and the integration measure of the lattice formulation of Ref. [16] are manifestly invariant under a lattice counterpart of a part of the  $\mathcal{N} = (2, 2)$  supersymmetry transformations,  $Q$ . By a similar reasoning as above, a hamiltonian density  $\mathcal{H}(x)$  is then defined by  $\mathcal{H}(x) \equiv Q\mathcal{J}_0^0(x)/2$ , where  $\mathcal{J}_0^0(x)$  is a lattice transcription of the Noether current associated with another fermionic symmetry of the target continuum theory,  $Q_0$ . This definition is consistent with the supersymmetry algebra in the twisted spinor basis,  $\{Q, Q_0\} = 2i\partial_0$ . From the  $Q$ -invariance of the lattice action and of the integration measure, we have  $\int_{\text{PBC}} d\mu \mathcal{H}(x) e^{-S} = 0$  (assuming that the integral  $\int_{\text{PBC}} d\mu \mathcal{J}_0^0(x) e^{-S}$  is finite) that is analogous to Eq. (5). Then the ground-state (vacuum) energy density  $\mathcal{E}_0$  is given by  $\lim_{\beta \rightarrow \infty} \lim_{a \rightarrow 0} \langle \mathcal{H}(x) \rangle_{\text{aPBC}} = \mathcal{E}_0$ . We judge that the dynamical supersymmetry breaking takes place if  $\mathcal{E}_0 > 0$  and it does not if  $\mathcal{E}_0 = 0$ .

Our algorithm and the simulation code, that were developed by using FermiQCD/MDP [19, 20], are almost identical to those of Ref. [21]. We use the hybrid Monte Carlo algorithm to generate configurations in the quenched approximation. The effect of dynamical fermions is then afterward taken into account by reweighting configurations by the pfaffian of the Dirac operator (we do not introduce any mass terms of fermions

or bosons that would explicitly break the  $Q$ -symmetry). Although this is certainly a brute force method compared to a standard pseudo-fermion algorithm, its implementation is much simpler and the validity has been observed for one-point Ward-Takahashi identities [21].

The number of statistically-independent configurations we used is summarized in Table 1, where  $N_T$  and  $N_S$  are the number of lattice points for the temporal and the spatial directions, respectively. The physical size of the spatial direction is fixed to be  $Lg = \sqrt{2}$ . We used the cold start and set all scalar fields to be zero at the initial configuration. As initial thermalization, we discarded the first  $10^4$  trajectories and then we stored configurations at each  $10^2$  trajectories (the auto-correlation time was 10–20 trajectories).

The result of our Monte Carlo simulation is Fig. 3. The result was obtained, as Fig. 4, by an extrapolation to the continuum  $a = 0$  by a linear  $\chi^2$ -fit. In Fig. 3, we observe that  $\mathcal{E}_0$  is consistent with zero within the error. We regard this as an indication that supersymmetry is not dynamically broken in the two-dimensional  $\mathcal{N} = (2, 2)$  super Yang-Mills theory with the gauge group  $SU(2)$ . Of course, errors in our present result are still large and we cannot completely exclude a possibility of the supersymmetry breaking of  $O(1)$  in  $\mathcal{E}_0/g^2$ . Further reduction of statistical errors will allow us to conclude whether the scale of the dynamical supersymmetry breaking is  $O(1)$  or not [30]. If the above observation of unbroken supersymmetry is true, the expectation value of the

hamiltonian density under the periodic boundary condition  $\langle \mathcal{H}(x) \rangle_{\text{PBC}}$  should be well-defined and vanishes for all  $\beta$ . Actually, the real part of the expectation values  $\text{Re}\langle \mathcal{H}(x) \rangle_{\text{PBC}}$  for various  $\beta g$  we plotted in Fig. 3 show that this requirement is met within errors. (The imaginary part  $\text{Im}\langle \mathcal{H}(x) \rangle_{\text{PBC}}$  is also consistent with zero and the errors are quite smaller than those for the real part.) This provides another support for our observation.

To our knowledge, this work is the first instance that the dynamical supersymmetry breaking in a gauge field theory (for which the Witten index is not known) is investigated by numerical simulation. It should be interesting to consider applications of the present method to other supersymmetric theories and gain an insight on a possible supersymmetry breaking that is difficult to obtain in other ways.

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- [1] E. Witten, Nucl. Phys. B **202**, 253 (1982).
  - [2] D. B. Kaplan, Nucl. Phys. (Proc. Suppl.) **129**, 109 (2004); arXiv:hep-lat/0309099.
  - [3] A. Feo, Mod. Phys. Lett. A **19** (2004), 2387; arXiv:hep-lat/0410012.
  - [4] J. Giedt, Int. J. Mod. Phys. A **21**, 3039 (2006); arXiv:hep-lat/0602007.
  - [5] J. Giedt, PoS **LAT2006**, 008 (2006); arXiv:hep-lat/0701006.
  - [6] E. Witten, Nucl. Phys. B **188**, 513 (1981).
  - [7] S. Catterall and T. Wiseman, J. High Energy Phys. **12**, 104 (2007); arXiv:0706.3518 [hep-lat].
  - [8] K. N. Anagnostopoulos, M. Hanada, J. Nishimura and S. Takeuchi, Phys. Rev. Lett. **100**, 021601 (2008); arXiv:0707.4454 [hep-th].
  - [9] S. Cecotti and L. Girardello, Phys. Lett. B **110**, 39 (1982).
  - [10] K. Fujikawa, Z. Physik C **15**, 275 (1982).
  - [11] M. Beccaria, M. Camprostrini and A. Feo, Phys. Rev. D **69**, 095010 (2004); arXiv:hep-lat/0402007.
  - [12] M. Beccaria, G. Curci and E. D'Ambrosio, Phys. Rev. D **58**, 065009 (1998); arXiv:hep-lat/9804010.
  - [13] S. Catterall and E. Gregory, Phys. Lett. B **487**, 349 (2000); arXiv:hep-lat/0006013.
  - [14] S. Catterall, J. High Energy Phys. **05**, 038 (2003); arXiv:hep-lat/0301028.
  - [15] I. Kanamori, F. Sugino and H. Suzuki, arXiv:0711.2132 [hep-lat].
  - [16] F. Sugino, J. High Energy Phys. **03**, 067 (2004); arXiv:hep-lat/0401017.
  - [17] F. Sugino, J. High Energy Phys. **01**, 015 (2004); arXiv:hep-lat/0311021.
  - [18] K. Hori and D. Tong, J. High Energy Phys. **05**, 079 (2007); arXiv:hep-th/0609032.
  - [19] M. Di Pierro, Comput. Phys. Commun. **141**, 98 (2001); arXiv:hep-lat/0004007.
  - [20] M. Di Pierro and J. M. Flynn, PoS **LAT2005**, 104 (2006); arXiv:hep-lat/0509058.
  - [21] H. Suzuki, J. High Energy Phys. **09**, 052 (2007); arXiv:0706.1392 [hep-lat].
  - [22] The thermal average of the hamiltonian in  $(1+0)$ -dimensional supersymmetric gauge theories has been numerically investigated in Refs. [7, 8].
  - [23] Note however a crucial difference from ordinary symmetries; supersymmetry can be broken even with finite spatial volume [6].
  - [24] This is precisely the idea of the hamiltonian formulation of supersymmetric theories (see Ref. [11] and references cited therein). From a view point of the gauge invariance, however, the euclidean lattice formulation appears advantageous.
  - [25] Interestingly, in models we investigate, the additional terms cancel with each other and do not contribute to one-point functions [15].
  - [26] The result was obtained by an extrapolation to the continuum  $a = 0$  by a linear  $\chi^2$ -fit of data computed at

TABLE I: The number of statistically-independent configurations we used.

$N_S$	$ag$	$N_T/N_S$						
		0.25	0.5	1	1.5	2	2.5	3
6	0.2357	—	39,900	99,900	9,900	9,900	9,900	9,900
8	0.1768	—	39,900	99,900	9,900	9,900	9,900	9,900
12	0.1179	39,900	69,900	69,900	9,900	9,900	9,900	9,900
16	0.08839	39,900	—	—	—	—	—	—
20	0.07071	39,900	—	—	—	—	—	—

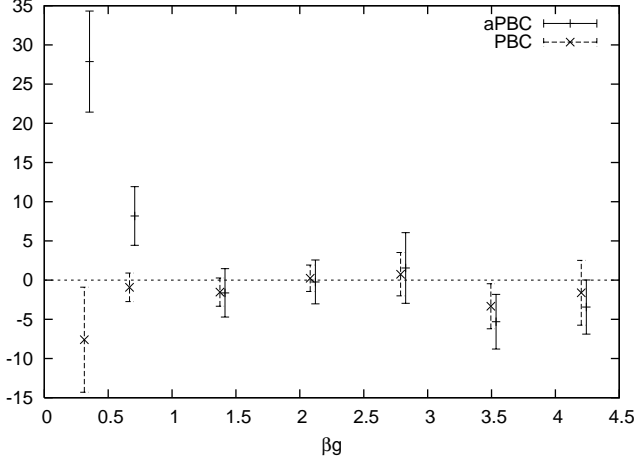


FIG. 3:  $3 \cdot \lim_{a \rightarrow 0} \text{Re}\langle \mathcal{H}(x) \rangle_{\text{aPBC}} / g^2$  and  $\lim_{a \rightarrow 0} \text{Re}\langle \mathcal{H}(x) \rangle_{\text{PBC}} / g^2$ , as a function of the physical temporal size of the system  $\beta g$ . The errors are only statistical ones.

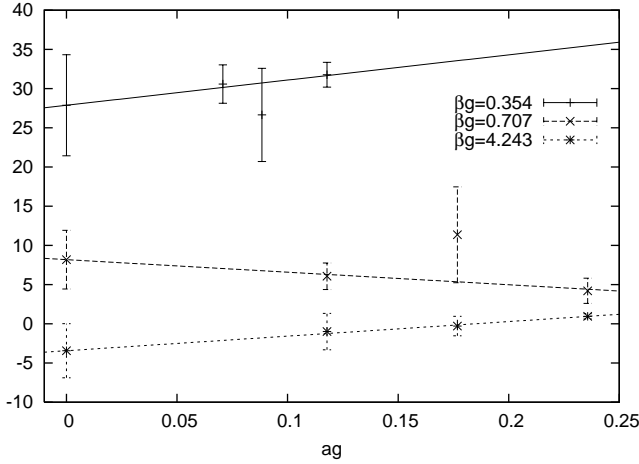


FIG. 4: Linear  $\chi^2$ -extrapolation of  $\text{Re}\langle \mathcal{H}(x) \rangle_{\text{aPBC}} / g^2$  to the continuum  $a = 0$  for various values of  $\beta g$ . The errors are only statistical ones.

$am = 0.1, 0.05$  and  $0.02$ . We used  $10^4$  statistically independent configurations for each set of parameters.

- [27] In Fig. 1, we plotted also the exact ground-state energy  $E_0/m = 1.27616$  that was obtained by a numerical diagonalization of the corresponding hamiltonian. The discrepancy of our Monte Carlo result for  $\beta m \gtrsim 1$  with this exact result can be understood by a systematic error associated with the linear extrapolation to the continuum.
- [28] The result was obtained by an extrapolation to the continuum  $a = 0$  by a linear  $\chi^2$ -fit of data computed at  $am = 0.1$  and  $0.05$ . We used  $10^4$  statistically-independent configurations for each set of parameters.
- [29] Space does not permit to reproduce relevant mathematical expressions and full details of the simulation. We refer the reader to Ref. [15] for these and for a further list of references.
- [30] For the present lattice model, we are at present developing a simulation code with the pseudo-fermion and the RHMC algorithm. We hope this enable us to reduce the statistical errors, without increasing the number of configurations substantially.